

Competition in Wireless Systems via Bayesian Interference Games

Sachin Adlakha, Ramesh Johari, and Andrea Goldsmith*

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Abstract

We study competition between wireless devices with *incomplete information* about their opponents. We model such interactions as *Bayesian interference games*. Each wireless device selects a power profile over the entire available bandwidth to maximize its data rate (measured via Shannon capacity), which requires mitigating the effect of interference caused by other devices. Such competitive models represent situations in which several wireless devices share spectrum without any central authority or coordinated protocol.

In contrast to games where devices have complete information about their opponents, we consider scenarios where the devices are unaware of the interference they cause to other devices. Such games, which are modeled as Bayesian games, can exhibit significantly different equilibria. We first consider a simple scenario where the devices select their power profile simultaneously. In such *simultaneous move games*, we show that the unique Bayes-Nash equilibrium is where both devices spread their power equally across the entire bandwidth. We then extend this model to a two-tiered spectrum sharing case where users act sequentially. Here one of the devices, called the *primary user*, is the owner of the spectrum and it selects its power profile first. The second device (called the *secondary user*) then responds by choosing a power profile to maximize its Shannon capacity. In such *sequential move games*, we show that there exist equilibria in which the primary user obtains a higher data rate by using only a part of the bandwidth.

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In a *repeated* Bayesian interference game, we show the existence of *reputation effects*: an informed primary user can “bluff” to prevent spectrum usage by a secondary user who suffers from lack of information about the channel gains. The resulting equilibrium can be highly inefficient, suggesting that competitive spectrum sharing is highly suboptimal. This observation points to the need for some regulatory protocol to attain a more efficient spectrum sharing solution.

1 Introduction

Our paper is motivated by a scenario where several wireless devices share the same spectrum. Such scenarios are a common occurrence in unlicensed bands such as the ISM and UNII bands. In such bands, diverse technologies such as 802.11, Bluetooth, Wireless USB, and cordless phones compete with each other for the same bandwidth. Usually, these devices have different objectives, they follow different protocols, and they do not cooperate with each other. Indeed, although the FCC is considering wider implementation of “open” spectrum sharing models, one potential undesirable outcome of open spectrum could be a form of the “tragedy of the commons”: self-interested wireless devices destructively interfere with each other, and thus eliminate potential benefits of open spectrum.

Non-cooperative game theory offers a natural framework to model such interactions between competing devices. In [16], the authors studied competition between devices in a Gaussian noise environment as a *Gaussian interference* (GI) game. This work was extended in [2] for the case of spectrum allocation between wireless devices; the authors provided a non-cooperative game theoretic framework to study issues such as spectral efficiency and fairness. In [9], the authors derived channel gain regimes where cooperative schemes would perform better than non-cooperative schemes for the GI game.

The game theoretic models used in these previous works typically assume that the matrix of channel gains among all users is completely known to the players. This may not be realistic or practical in many scenarios, as competing technologies typically do not employ a coordinated information dissemination protocol. Even if information dissemination protocols were employed, incentive mechanisms would be required in a situation with competitive devices to ensure that channel states were truthfully exchanged. By contrast, our paper studies a range of non-cooperative games characterized by the feature that there is *incomplete information* about some or all channel gains between devices. Such scenarios are captured through static and dynamic *Bayesian games* [3].

We consider a simplistic model where two transmitter-receiver (TX-RX) pairs, or “users”,

share a single band divided into K subchannels.¹ We assume both users face a total power constraint, and that the noise floor is identical across subchannels. We further assume that channel gains are drawn from a fixed distribution that is common knowledge to the users. We make the simplifying assumption of *flat fading*, i.e., constant gains across subchannels, to develop the model. A user’s strategic decision consists of an allocation of power across the available subchannels to maximize the available data rate (measured via Shannon capacity).

In Section 2 we consider a *simultaneous-move* game between the devices under this model. We study two scenarios: first, a game where all channel gains are unknown to both users; and second, a game where a user knows the gain between its own TX-RX pair as well as the interference power gain from the other transmitter at its own receiver (also called incident channel gains), but it does not know the channel gain between the TX-RX pair of the other user or the interference it causes to the other receiver. In these two scenarios, we show that there exists a unique symmetric Bayes-Nash equilibrium², where both users equally spread their power over the band (regardless of the channel gains observed). In this equilibrium, the actions played after channel gains are realized are also a Nash equilibrium of the complete information game.

While simultaneous-move games are a good model for competition between devices with equal priority to shared resources, they are not appropriate for a setting where one device is a natural incumbent, such as primary/secondary device competition. In such two-tiered models for spectrum sharing, some radio bands may be allocated to both primary and secondary users. The primary users have priority over the secondary users and we use game theory to analyze competition in such scenarios. In Section 3, we consider a two-stage *sequential* Bayesian game where one device (the primary) moves before the other (the secondary); we find that asymmetric equilibria can be sustained where the devices sometimes operate in disjoint subchannels (called “sharing” the bandwidth), provided interference between them is sufficiently large. We also add an *entry stage* to the game, where the secondary device decides whether or not it wants to operate in the primary’s band in the first place; we also characterize Nash equilibria of this game in terms of the distribution of the incident channel gains.

In Section 4, we use the sequential Bayesian game with entry to study *repeated* interaction between a primary and secondary user. We consider a model where a secondary user repeatedly polls a primary user’s band to determine if it is worthwhile to enter. Using techniques

¹Throughout this paper, a transmitter-receiver pair is identified with a particular user.

²Here “symmetric” means that both users’ strategies are identical functions of their channel gain. Asymmetric equilibria, where users may have different functional form of their strategies, are harder to justify, as they would require prior coordination among the devices to “agree” on which equilibrium is played.

pioneered in the economics literature on *reputation effects* [10], we show the existence of a *sequential equilibrium* where the primary user exploits the secondary user’s lack of channel knowledge to its own advantage; in particular, we show that by threatening to aggressively spread power against the secondary, the primary can deter the secondary from entering at all. (See footnote 4 for the definition of sequential equilibrium.) In a complete information game, the secondary user knows that the best response of the primary user to an entry by the secondary user is to share the bandwidth. Thus, the primary user’s threat of aggressively spreading power would not be *credible* in such scenarios. Our result suggests that, in the absence of regulation, primary devices may inflate their power profile to “scare” secondary devices away, even if such behavior is suboptimal for the primary in the short term.

We conclude by noting that game theoretic models have also been used in the *design* of power control and spectrum sharing schemes. A market-based power control mechanism for wireless data networks was discussed in [13]. In [1], the authors model a power control mechanism as a supermodular game and prove several convergence properties. Supermodularity was also employed to describe a distributed power control mechanism in [5]; this latter paper also contains insights regarding supermodularity of the GI game. Another approach to spectrum sharing is to consider real time “auctions” of the channels as described in [4, 14]. By contrast, our paper studies a setting where no coordination mechanism exists, and the devices are completely competitive; hence we do not follow this approach.

2 Static Gaussian Interference Games

In this section we consider a range of static game-theoretic models for competition between two devices. In other words, in all the models we consider, both devices *simultaneously* choose their actions, and then payoffs are realized. We start in Section 2.1 by defining the model we consider, an interference model with two users. In Section 2.2 we define a Bayesian game where both users do not know any of the channel gains. However, this model is not necessarily realistic; in many scenarios information is *asymmetric*: a device may know its own incident channel gains, but not those incident on the other devices. Thus in Section 2.3, we introduce a “partial” Bayesian game (where the users know their own channel gain as well as the received interference gain), and study its equilibria.

2.1 Preliminaries

We consider a two user Gaussian interference model (see Figure 1) with K subchannels; in each subchannel $k = 1, \dots, K$, the model is:

$$y_i[n] = \sum_{j=1}^2 h_{ji}^k x_j[n] + w_i[n], \quad i = 1, 2, \quad (1)$$

where $x_i[n]$ and $y_i[n]$ are user i 's input and output symbols at time n , respectively. Here h_{ij}^k is the channel gain from the transmitter of user i to the receiver of user j in subchannel k . We assume that the system exhibits *flat fading*, i.e., the channel gains $h_{ij}^k = h_{ij}$ for all $k = 1, \dots, K$. The noise processes $w_1[n]$ and $w_2[n]$ are assumed to be independent of each other, and are i.i.d over time with $w_i[n] \sim \mathcal{N}(0, N_0)$, where N_0 is the noise power spectral density.

Each user has an average power constraint of P . We assume that each user treats interference as noise and that no interference cancellation techniques are used. Denote by P_{ik} the transmission power of user i in channel k . Let $\mathbf{P}_i = (P_{i1}, \dots, P_{iK})$, and $\mathbf{P} = (\mathbf{P}_1, \mathbf{P}_2)$. We will frequently use the notation “ $-i$ ” to denote the player other than i (i.e., player 1 if $i = 2$, and player 2 if $i = 1$). The *utility* $\Pi_i(\mathbf{P})$ of user i is the Shannon capacity data rate limit for the user. Under the above assumptions, given a power vector \mathbf{P} , the Shannon capacity limit of a user i over all K subchannels is given as:

$$\Pi_i(\mathbf{P}) = \sum_{k=1}^K \left[\frac{1}{2} \log \left(1 + \frac{g_{ii} P_{ik}}{N_0 + g_{-i,i} P_{ik}} \right) \right]. \quad (2)$$

Here g_{ij} is the interference gain between the transmitter of user i and the receiver of user j , and is defined as $g_{ij} = |h_{ij}|^2$; we let $\mathbf{g} = (g_{11}, g_{12}, g_{21}, g_{22})$ denote the channel gain vector. Note that for each i , the power allocation must satisfy the constraint $\sum_k P_{ik} \leq P$. In particular, both users share the same power constraint.

In the complete information Gaussian interference (GI) game, each user i chooses a power allocation \mathbf{P}_i to maximize the utility $\Pi_i(\mathbf{P}_i, \mathbf{P}_{-i})$ subject to the total power constraint, given the power allocation \mathbf{P}_{-i} of the opponent. Both users carry out this maximization with full knowledge of the channel gains \mathbf{g} , the noise level N_0 , and the power limit P . A *Nash equilibrium* (NE) of this game is a power vector \mathbf{P} where both users have simultaneously maximized payoffs. This interference game has been analyzed previously in the literature, and in particular existence and conditions for uniqueness of the equilibrium have been developed in [2, 16].

In this paper, we take a different approach: we consider the same game, but assume some

or all of the channel gains are *unknown* to the players. In the next two sections, we introduce two variations on this game.

2.2 The Gaussian Interference Game with Unknown Channel Gains

We begin by considering the GI game, but where neither player has knowledge of the channel gains g_{ij} ; we refer to this as the *unknown channel GI (UC-GI) game*. Our motivation is the fast-fading scenario where the channel gains change rapidly relative to the transmission strategy decision. This makes the channel gain feedback computationally expensive and generally inaccurate.

We assume that the channel gains \mathbf{g} are drawn from a distribution F , with continuous density f on a compact subset $G \subset \{\mathbf{g} : g_{ij} > 0 \forall i, j\}$, and we assume that both players do not observe the channel gains. For simplicity, we assume that F factors so that (g_{11}, g_{21}) is independent of (g_{22}, g_{12}) .

We assume that both players now maximize *expected* utility, given the power allocation of their opponent; i.e., given \mathbf{P}_{-i} , player i chooses \mathbf{P}_i to maximize $\mathbb{E}[\Pi_i(\mathbf{P}_i, \mathbf{P}_{-i})]$, subject to the power constraint (the expectation is taken over the distribution F). A NE of the UC-GI game is thus a power vector where both players have simultaneously maximized their expected payoffs.

We focus our attention on the case of *symmetric* NE, i.e., where both players use the same strategy. It is possible that there may exist several asymmetric equilibria, but for the users to operate at any one of those equilibria would require some form of prior coordination. Since the users in this game do not coordinate, it is reasonable to search for symmetric equilibrium. The next theorem shows that if $K = 2$, the UC-GI game has a unique symmetric Nash equilibrium.

Theorem 1. *For the UC-GI game with $K = 2$ subchannels, there exists a unique symmetric pure strategy Nash equilibrium, regardless of the channel distribution F , where the users spread their power equally over the entire band; i.e., the unique NE is $P_{11}^* = P_{12}^* = P_{21}^* = P_{22}^* = P/2$.*

Proof. Note that if \mathbf{P}^* is a NE, then (substituting the power constraint) we conclude P_{i1}^* is a solution of the following maximization problem:

$$\max_{P_{i1}} \int_G \left[\frac{1}{2} \log \left(1 + \frac{g_{ii}P_{i1}}{N_0 + g_{-i,i}P_{-i,1}} \right) + \frac{1}{2} \log \left(1 + \frac{g_{ii}(P - P_{i1})}{N_0 + g_{-i,i}(P - P_{-i,1})} \right) \right] f(\mathbf{g}) d\mathbf{g}.$$

Since $\log(1+x)$ is strictly concave in x , the first order conditions are necessary and sufficient to identify a NE. Differentiating and simplifying yields:

$$\int_G \frac{g_{ii}}{2} \left(\frac{g_{ii}(P - 2P_{i1}) + g_{-i,i}(P - 2P_{-i,1})}{(N_0 + g_{ii}P_{i1} + g_{-i,i}P_{-i,1})(N_0 + g_{ii}(P - P_{i1} + g_{-i,i}(P - P_{-i,1})))} \right) f(\mathbf{g}) d\mathbf{g} = 0.$$

Note that the denominator in the integral above is always positive; and further, $g_{ii} > 0$ on G . Thus in a NE, if $P_{i1} > P/2$, then we must have $P_{-i,1} < P/2$ (and vice versa). Thus the only symmetric NE occur where $P_{i1} = P_{-i,1} = P/2$, as required. \square

While our result is framed with only two subchannels, the same argument can be easily extended to the case of multiple subchannels via induction.

Corollary 1. *Consider the UC-GI game with $K > 1$ subchannels. There exists a unique symmetric NE, where the two users spread their power equally over all K subchannels.*

Proof. The proof follows from an inductive argument; clearly the result holds if $K = 2$. Let \mathbf{P}^K be a symmetric NE with K subchannels. Let $S \subset \{1, \dots, K\}$ be a subset of the subchannels. Since the NE is symmetric, let $Q^S = \sum_{k \in S} P_{ik}^K$; this is the total power the players use in the subchannels of S . It is clear that if we *restrict* the power vector \mathbf{P}^K to only the subchannels in S , then the resulting power vector must be a symmetric NE for the UC-GI game over only these subchannels, with total power constraint Q^{K-1} . Since this holds for every subset $S \subset \{1, \dots, K\}$ of size $|S| \leq K - 1$, we can apply the inductive hypothesis to conclude every user allocates equal power to each subchannel in the equilibrium \mathbf{P}^K , as required. \square

While we have only shown uniqueness among *symmetric* NE in the preceding results, we conjecture that in fact the *only* pure NE of the UC-GI game is one where all players use equal transmit power in every subchannel. Our conjecture is motivated by numerical results using *best response dynamics* for the UC-GI game; these are dynamics where at each time step, each player plays a best response to the action of his opponent at the previous time step. As we see in Figure 2, even if the users initially transmit at different powers in each subchannel, the best response dynamics converge to the symmetric NE. In fact, for this numerical example the best response dynamics verify uniqueness of the pure NE.³

³For the UC-GI game, one can infer that for the numerical example with $K = 2$, the unique pure NE is the symmetric NE where all users spread their power across the subchannels. To justify this claim, note that the UC-GI game is a *supermodular game*, if the strategy spaces are appropriately defined. (A complete overview of supermodular games is beyond the scope of this paper; for background on supermodular games, see [5, 15].) In particular, let $s_1 = P_{11}$, and let $s_2 = -P_{21}$, with strategy spaces $S_1 = [0, P]$, $S_2 = [-P, 0]$. Define $V_i(s_1, s_2) = \Pi_i(s_1, P - s_1, -s_2, P + s_2)$. Then it can be easily shown that V_i has increasing differences

2.3 Bayesian Gaussian Interference Game

In the UC-GI game defined above, we assume that each user is unaware of all the channel gains. However in a slowly changing environment, it is common for the receiver to feed back channel gain information to the transmitter. Thus, in this section we assume that each user i is aware of the self channel gain g_{ii} , and the incident channel gain $g_{-i,i}$; for notational simplicity, let $\mathbf{g}_i = (g_{ii}, g_{-i,i})$. However, because of the difficulties involved in dissemination of channel state information from other devices, we continue to assume that each user is unaware of the channel gains of the other users. In particular, this means user i does not know the value \mathbf{g}_{-i} . In this game, each player chooses \mathbf{P}_i to maximize $\mathbb{E}[\Pi_i(\mathbf{P}_i, \mathbf{P}_{-i})|\mathbf{g}_i]$, subject to the power constraint; note that now the expectation is conditioned on \mathbf{g}_i . The power allocation \mathbf{P}_{-i} is random, since it depends on the channel gains of player $-i$ —which are unknown to player i . Thus this is a *Bayesian game*, in which a strategy of player i is a family of functions $\mathbf{s}_i(\mathbf{g}_i) = (s_{i1}(\mathbf{g}_i), \dots, s_{iK}(\mathbf{g}_i))$, where $s_{ik}(\mathbf{g}_i)$ gives the power allocation of player i in subchannel k when gains \mathbf{g}_i are realized. We refer to this game as the *Bayesian Gaussian interference (BGI) game*. A *Bayes-Nash equilibrium* BNE is a strategy vector $(\mathbf{s}_1(\cdot), \mathbf{s}_2(\cdot))$ such that for each i and each \mathbf{g}_i , player i has maximized his expected payoff given the strategy of the opponent:

$$\mathbf{s}_i(\mathbf{g}_i) \in \arg \max_{\mathbf{P}_i} \mathbb{E}[\Pi_i(\mathbf{P}_i, \mathbf{s}_{-i}(\mathbf{g}_{-i}))|\mathbf{g}_i].$$

For the BGI game, we again want to investigate symmetric Bayes-Nash equilibria. However, in principle the functional strategic form of a player can be quite complex. Thus, for analytical tractability, we focus our attention on a restricted class of possible actions: we allow users to either put their entire power in a single subchannel, or split their power evenly across all subchannels. This is a practical subclass of actions which allows us to explore whether asymmetric equilibria can exist.

Formally, the action space of both players is now restricted to $S = \{P\mathbf{e}_1, \dots, P\mathbf{e}_K, P\mathbf{1}/K\}$; here \mathbf{e}_i is the standard basis vector with all zero entries except a “1” in the i ’th position, and $\mathbf{1}$ is a vector where every entry is “1”. Thus $P\mathbf{e}_k$ is the action that places all power in subchannel k , while $P\mathbf{1}/K$ spreads power equally across all subchannels. A strategy for player i is a map that chooses, for each realization of $(g_{ii}, g_{-i,i})$, an action in S .

in s_i and s_{-i} . This suffices to ensure that there exists a “largest” NE $\bar{\mathbf{s}}$, and a “smallest” NE $\underline{\mathbf{s}}$, that are, respectively, the least upper bound and greatest lower bound to the set of NE in the product lattice $S_1 \times S_2$ [12]. Further, best response dynamics initiated at the smallest strategy vector $(s_1, s_2) = (0, -P)$ converge to $\underline{\mathbf{s}}$; and best response dynamics initiated at the largest strategy vector $(s_1, s_2) = (P, 0)$ converge to $\bar{\mathbf{s}}$ [12]. Thus if these two best response dynamics converge to the same strategy vector, there must be a unique pure NE.

Our main result is the following theorem.

Theorem 2. *Assume that (g_{11}, g_{21}) and (g_{22}, g_{12}) are i.i.d. Then the unique pure strategy symmetric BNE of the BGI game is where both users choose action $P\mathbf{1}/K$, i.e., they spread their power equally across bands.*

Proof. Fix a symmetric BNE $(\mathbf{s}_1, \mathbf{s}_2)$ where $s_{1k}(\cdot) = s_{2k}(\cdot) = s^k(\cdot)$ is the common strategy used by both players; i.e., given channel gains \mathbf{g}_i , player i puts power $s^k(\mathbf{g}_i)$ in subchannel k . Define $\alpha_k = \mathcal{P}(s^k(\mathbf{g}_i) = P\mathbf{e}_k)$ for each subchannel k ; and $\gamma = \mathcal{P}(s^k(\mathbf{g}_i) = P\mathbf{1}/K)$. These are the probabilities that a player transmits with full power in subchannel k , or with equal power in all subchannels, respectively.

Let $\bar{\Pi}_i(\mathbf{P}_i; \mathbf{g}_i)$ be the expected payoff of user 1 if it uses action $\mathbf{P}_i \in S$, given that the other player is using the equilibrium strategy profile (s^1, \dots, s^K) and the channel gains are \mathbf{g}_i . We start with the following lemma.

Lemma 3. *For two subchannels k, k' , if $\alpha_k < \alpha_{k'}$, then $\bar{\Pi}_i(P\mathbf{e}_k; \mathbf{g}_i) > \bar{\Pi}_i(P\mathbf{e}_{k'}; \mathbf{g}_i)$ for all values of \mathbf{g}_i ; i.e., player i strictly prefers to put full power into subchannel k over putting full power into subchannel k' .*

Proof of Lemma. Using (2) we can write $\bar{\Pi}_i(P\mathbf{e}_k; \mathbf{g}_i)$ as:

$$\begin{aligned} \bar{\Pi}_i(P\mathbf{e}_k; \mathbf{g}_i) &= \frac{\alpha_k}{2} \left[\log \left(1 + \frac{g_{ii}P}{N_0 + g_{-i,i}P} \right) \right] + \frac{\gamma}{2} \left[\log \left(1 + \frac{g_{ii}P}{N_0 + g_{-i,i}P/B} \right) \right] \\ &\quad + \frac{1 - \alpha_k - \gamma}{2} \left[\log \left(1 + \frac{g_{ii}P}{N_0} \right) \right]. \end{aligned}$$

Define Δ as:

$$\Delta \triangleq \frac{1}{2} \log \left(1 + \frac{g_{ii}P}{N_0} \right) - \frac{1}{2} \log \left(1 + \frac{g_{ii}P}{N_0 + g_{-i,i}P} \right) > 0,$$

since we have assumed $g_{-i,i} > 0$. Rearranging and simplifying, we have that $\bar{\Pi}_i(P\mathbf{e}_k; \mathbf{g}_i) - \bar{\Pi}_i(P\mathbf{e}_{k'}; \mathbf{g}_i) = \Delta (\alpha_{k'} - \alpha_k)$. Since $\Delta > 0$ and $\alpha_{k'} > \alpha_k$, the lemma is proved. \square

The previous lemma ensures that in a symmetric equilibrium we cannot have $\alpha_k < \alpha_{k'}$ for any two subchannels k, k' : in this case, $\alpha_{k'} > 0$, so the equilibrium strategy puts positive weight on action $P\mathbf{e}_{k'}$; but player i 's best response to this strategy puts zero weight on $P\mathbf{e}_{k'}$ (from the lemma).

Thus in a symmetric equilibrium we must have $\alpha_k = \alpha_{k'}$ for all subchannels k, k' ; i.e.,

$\alpha_k = (1 - \gamma)/K$ for all k . Define $\alpha = (1 - \gamma)/K$. In this case we have for each subchannel k :

$$\begin{aligned}\bar{\Pi}_i(P\mathbf{e}_k; \mathbf{g}_i) = & \alpha \left[\frac{K-1}{2} \log \left(1 + \frac{g_{ii}P}{N_0} \right) + \frac{1}{2} \log \left(1 + \frac{g_{ii}P}{N_0 + g_{-i,i}P} \right) \right] \\ & + \gamma \left[\frac{1}{2} \log \left(1 + \frac{g_{ii}P}{N_0 + g_{-i,i}P/K} \right) \right],\end{aligned}$$

and

$$\begin{aligned}\bar{\Pi}_i(P\mathbf{1}/K) = & (1 - \gamma) \left[\frac{K-1}{2} \log \left(1 + \frac{g_{ii}P/K}{N_0} \right) + \frac{1}{2} \log \left(1 + \frac{g_{ii}P/K}{N_0 + g_{-i,i}P} \right) \right] \\ & + \gamma \left[\frac{K}{2} \log \left(1 + \frac{g_{ii}P/K}{N_0 + g_{-i,i}P/K} \right) \right].\end{aligned}$$

Since $\log(1+x)$ is a strictly concave function of x , we get that $K \log(1 + \frac{x}{K}) > \log(1+x)$ for $x > 0$. Since $\alpha = (1 - \gamma)/2$, this implies that $\bar{\Pi}_i(P\mathbf{1}/K) > \bar{\Pi}_i(P\mathbf{e}_k)$ for all subchannels k . Thus in a symmetric equilibrium, we must have $\alpha = 0$; i.e., the unique symmetric equilibrium occurs where $\gamma = 1$, so both users equally spread their power across all subchannels. \square

Thus far we have considered games where players act simultaneously. However in several practical cases, one of the players may already be using the spectrum when another user wants to enter the same band. We model such scenarios as sequential games in the next section.

3 Sequential Interference Games with Incomplete Information

In this section, we study sequential games between wireless devices with incomplete information. In such games, player 1, who we refer to as the *primary user*, determines its transmission strategy before player 2. Player 2, also referred to as the *secondary user*, observes the action of the primary user and chooses a transmission strategy that is a best response. We study this model in Section 3.1. While we focused on symmetric equilibria of static games in the preceding section, we focus here on the fact that sequential games naturally allow the users to sustain asymmetric equilibrium. We characterize how these equilibria depend on the realized channel gains of the users.

Such games are a natural approach to study dynamic spectrum sharing between cognitive radios. The primary user is the incumbent user of the band, while the secondary user represents a potential new device that also wishes to use spectrum in the band. In particular,

in this model we must also study *whether* the secondary device would find it profitable to compete with the primary in the first place. In Section 3.2, we extend this game to incorporate an entry decision by the secondary user, and again study dependence of the equilibria on realized channel gains.

3.1 A Two-Stage Sequential Game

In this section, we consider a two-stage *sequential Bayesian Gaussian interference (SBGI)* game; we restrict attention to two subchannels for simplicity. Player 1 (the primary) moves first, and Player 2 (the secondary) *perfectly* observes the action of the primary user before choosing its own action. We assume that each of the self gains g_{ii} (in the interference channel given in (1)) are normalized to 1; this is merely done to isolate and understand the effects of interference on the interaction between devices, and does not significantly constrain the results.

As before, the channel gains are randomly selected. For the remainder of this section, we make the following assumption.

Assumption 1. *Player 1 (the primary user) knows both g_{12} and g_{21} , while the secondary user only knows g_{12} (but not g_{21}).*

Thus the primary user knows the interference it causes to the secondary user; the secondary user however, is only aware of its own incident channel gain. This approach allows us to focus on the secondary user’s uncertainty; it is also possible to analyze the same game when the primary user does not know g_{12} (see appendix).

As before, we restrict the action space of each user to either use only one of the subchannels, or to spread power equally in both subchannels. We assume that if the primary concentrates power in a single subchannel, it chooses subchannel 1; this is done without loss of generality, since fading is flat, and the primary moves first. If the primary user chooses to place its entire power in subchannel 1, then from Lemma 3, we know that the secondary user’s best response puts zero weight on this action: the secondary user will either spread over both subchannels, or put its entire power in the free subchannel. Thus concentrating power in a single subchannel is tantamount to *sharing* a single subchannel with the secondary. Thus we say the primary “shares” if it concentrates all its power in a single subchannel, and denote this action by SH . We use SP to denote the action where the primary spreads its power across both subchannels. We also use the same notation to denote the actions of the secondary: concentrating power in a single subchannel (in this case, subchannel 2) is denoted

by SH , and spreading is denoted by SP . The game tree for the SBGI game is shown in Figure 3.

We solve for the equilibrium path in the sequential game using backward induction. Once channel gains are realized, suppose that the primary player chooses the action SP . Conditioned on this action by the primary player, the secondary player chooses the action SP if

$$\log \left(1 + \frac{P/2}{N_0 + g_{12}P/2} \right) > \frac{1}{2} \log \left(1 + \frac{P}{N_0 + g_{12}P/2} \right). \quad (3)$$

Since $\log(1+x)$ is a strictly concave function of x , the above inequality holds for all values of g_{12} . Thus the best response of the secondary user is to choose SP whenever the primary user chooses SP , regardless of the value of g_{12} .

The situation is more interesting if the primary user decides to share the bandwidth, i.e., chooses action SH . Given this action of the primary user, the secondary user would prefer to spread its power if and only if:

$$\frac{1}{2} \log \left(1 + \frac{P}{2N_0} \right) + \frac{1}{2} \log \left(1 + \frac{P/2}{N_0 + g_{12}P} \right) > \frac{1}{2} \log \left(1 + \frac{P}{N_0} \right) \iff g_{12} < \frac{1}{2}. \quad (4)$$

Thus if $g_{12} < 1/2$, the secondary will choose SP in response to SH . On the other hand, if $g_{12} > 1/2$, the secondary user will choose SH in response to SH . (We ignore $g_{12} = 1/2$ since channel gains have continuous densities.) We summarize our observations in the next lemma.

Lemma 4. *Suppose Assumption 1 holds. In the SBGI game, if the primary user chooses SP in the first stage, the best response of the secondary is SP ; if the primary chooses SH in the first stage, the best response of the secondary is SH if $g_{12} > 1/2$, and SP if $g_{12} < 1/2$.*

Thus in equilibrium, regardless of the channel gains, either both users share or both users spread. Suppose the secondary plays SP regardless of the primary's action; in this case SP is also the best action for the primary. Thus if $g_{12} < 1/2$, the primary user will never choose SH in the first stage. On the other hand, when $g_{12} > 1/2$, it can choose its optimal action by comparing the payoff when *both* players choose SP to the payoff when *both* players choose SH . In this case, the primary user would prefer to spread its power if and only if:

$$\log \left(1 + \frac{P/2}{N_0 + g_{21}P/2} \right) > \frac{1}{2} \log \left(1 + \frac{P}{N_0} \right), \iff g_{21} < \frac{1}{\sqrt{1 + P/N_0} - 1} - \frac{2N_0}{P} \triangleq g^*. \quad (5)$$

Observe that this threshold approaches zero as $P/N_0 \rightarrow \infty$, and $1/2$ as $P/N_0 \rightarrow 0$. (In fact, the threshold is a decreasing function of P/N_0 .) Thus we have the following proposition.

Proposition 1. *Suppose Assumption 1 holds. In the sequential equilibria of the game,⁴ if $g_{12} > 1/2$ and $g_{21} > g^*$, the primary user chooses SH; if $g_{12} < 1/2$ or $g_{21} < g^*$, the primary user chooses SP. The secondary user plays the same action as that chosen by the primary, regardless of the realized channel gain. The value of g^* is given in (5).*

Since in equilibrium, either both users share or both users spread, for later reference we make the following definitions:

$$\Pi_i^{\text{share}} = \frac{1}{2} \log \left(1 + \frac{P}{N_0} \right); \quad \Pi_i^{\text{spread}} = \log \left(1 + \frac{P/2}{N_0 + g_{-i,i}P/2} \right). \quad (6)$$

These are the payoffs to user i if *both* users play SH, or both play SP, respectively. Note that Π_i^{spread} depends on $g_{-i,i}$, and is thus stochastic.

3.2 A Sequential Game with Entry

In this section, we modify the game of the last section to incorporate an entry decision by the secondary user; we refer to this as the *sequential Bayesian Gaussian interference game with entry (SBGI-E)*. Specifically, at the beginning of the game, the secondary user decides to either enter the system (this action is denoted by N) or stay out of the system (denoted by X). If the secondary user decides to exit, the primary user has no action to take. If however, the secondary user enters the system, the two users play the same game as described in the previous section. The game tree of the SBGI-E game is shown in Figure 3.

If the secondary user exits the game, its payoff (defined as the maximum data rate received) is 0. However, if the secondary user enters the game, the payoff to the user is always positive, regardless of the action taken by the primary user and the channel gains. Thus, in the absence of any further assumption, the model trivially reduces to the one

⁴Sequential equilibrium is a standard solution concept for dynamic games of incomplete information [8]. Sequential equilibrium consists of two elements for each user: a history- and type-dependent strategy, as well as a conditional distribution (or *belief*) over the unknown types of other players given history. Two conditions must be satisfied; first, for each player, the strategy maximizes expected payoff over the remainder of the game (i.e., the strategy is *sequentially rational*). Second, the beliefs are *consistent*: players compare observed history to the equilibrium strategies, and use Bayesian updating to specify their conditional distribution over other players' types. The precise definition is somewhat more involved, and beyond the scope of this paper.

In our scenario, the primary knows g_{12} , and thus has no uncertainty. The secondary user's belief of the value of g_{21} is updated at the second stage on the basis of the primary's action at the first stage; however, since the secondary user's action does not depend on the value of g_{21} , we do not specify beliefs in the statement of the proposition.

studied in the previous section. To make the model richer, we introduce a *cost of power* to the overall payoff of the secondary user.⁵ This cost of power can represent, for example, battery constraints of the wireless device. We assume that if the secondary enters, a cost kP is incurred (where k is a proportionality constant). With this cost of power, if the secondary user enters it obtains payoff $\hat{\Pi}_2(\mathbf{P}_1, \mathbf{P}_2) = \Pi_2(\mathbf{P}_1, \mathbf{P}_2) - kP$. Note that if the secondary user exits the game, it gets no rate with no cost of power; in this case its payoff is zero. Furthermore, from Proposition 1, we know that if the secondary user enters in equilibrium, it will obtain either $\Pi_2^{\text{share}} - kP$ or $\Pi_2^{\text{spread}} - kP$. Thus, to decide between entry and exit, the secondary user compares these quantities to zero. In the case where $g_{12} < 1/2$, we easily obtain the following proposition.

Proposition 2. *Suppose Assumption 1 holds, and that $g_{12} < 1/2$. In the sequential equilibrium of the SBGI-E game, the secondary player always enters if $\Pi_2^{\text{spread}} > kP$ and it always exits if $\Pi_2^{\text{spread}} < kP$.*

Proof. The proof follows trivially from Theorem 1 since if $g_{12} < 1/2$, both the primary and the secondary user spread their powers after entry. \square

Let us now consider the case when $g_{12} > 1/2$. Since the game is symmetric, we conclude that $\Pi_2^{\text{share}} > \Pi_2^{\text{spread}}$ if and only if $g_{12} > g^*$. A straightforward calculation using the expression in (5) establishes that $g^* < 1/2$. Thus if $g_{12} > 1/2$, then $g_{12} > g^*$; in particular, for $g_{12} > 1/2$, there always holds $\Pi_2^{\text{share}} > \Pi_2^{\text{spread}}$. For the sake of simplicity, we make the following assumption for the remainder of the paper.

Assumption 2. *The payoff to the secondary user Π_2^{share} is greater than the cost of power kP ; i.e., $P/N_0 > 2^{2kP} - 1$.*

We now compare Π_2^{spread} to the cost of power kP . Under Assumption 2, the secondary user would always enter if $\Pi_2^{\text{spread}} > kP$. This happens if and only if:

$$\log \left(1 + \frac{P/2}{N_0 + g_{12}P/2} \right) > kP \iff g_{12} < \frac{1}{2^{kP} - 1} - \frac{2N_0}{P} \triangleq \tilde{g}_{12}. \quad (7)$$

Thus if $1/2 < g_{12} < \tilde{g}_{12}$, the secondary user always enters. Note that $\tilde{g}_{12} \rightarrow -\infty$ if $N_0 \rightarrow \infty$, for fixed P . For fixed N_0 , we have $\tilde{g}_{12} \rightarrow 0$ as $P \rightarrow \infty$; and a straightforward calculation⁶

⁵This cost of power can also be introduced for the primary user without changing any results mentioned in this paper. We avoid this for the sake of simplicity.

⁶To see this, let $c = 2kN_0 \ln 2$ and $x = P/(2N_0)$, and note that:

$$\tilde{g}_{12} = \frac{1}{e^{cx} - 1} - \frac{1}{x}.$$

shows that:

$$\lim_{P \rightarrow 0} \tilde{g}_{12} = \begin{cases} \infty, & \text{if } 2kN_0 \ln 2 < 1; \\ -\infty, & \text{if } 2kN_0 \ln 2 > 1; \\ -1/2, & \text{if } 2kN_0 \ln 2 = 1. \end{cases}$$

In particular, \tilde{g}_{12} can take any real value depending on the parameters of the game.

If $g_{12} > \tilde{g}_{12}$, then $\Pi_2^{\text{spread}} < kP < \Pi_2^{\text{share}}$, so the secondary would only enter if the primary user shares the channel. Let $\rho = \mathcal{P}(g_{21} < g^*)$ be the probability that the primary user spreads the power. Then the expected payoff of the secondary user on entry is $\bar{\Pi}_2 = \rho \Pi_2^{\text{spread}} + (1 - \rho) \Pi_2^{\text{share}} - kP$. The secondary user would enter if its expected payoff is positive. This happens if and only if:

$$\rho < \frac{\Pi_2^{\text{share}} - kP}{\Pi_2^{\text{share}} - \Pi_2^{\text{spread}}} \triangleq d. \quad (8)$$

The equilibria of the SBGI-E for $g_{12} > 1/2$ are summarized in the following proposition.

Proposition 3. *Suppose Assumptions 1 and 2 hold, and that $g_{12} > 1/2$. Define d as in (8), and $\rho = \mathcal{P}(g_{21} < g^*)$. Then in the sequential equilibrium of the game, if $g_{12} < \tilde{g}_{12}$, the secondary user always enters the game; if however $g_{12} > \tilde{g}_{12}$ the secondary user enters the game if $\rho < d$, and it exits the game if $\rho > d$. Upon entry the primary and the secondary user follow the sequential equilibrium of the SBGI game as given in Proposition 1.*

4 Repeated Games with Entry: The Reputation Effect

In this section, we study *repeated* interactions between wireless devices with incomplete information about their opponents. We consider a finite horizon repeated game, where in each period the primary and secondary users play the SBGI-E game studied in the previous section. Consider, for example, a single secondary device considering “entering” one of several distinct bands, each owned by a different primary. The secondary is likely to *poll* the respective bands of the primaries, probing to see if entering the band is likely to yield a high data rate. Each time the secondary probes a single primary user’s band, it effectively decides whether to enter or exit; we model each such stage as the SBGI-E game of the preceding section.

For analytical simplicity, we assume that the secondary user is *myopic*; i.e., it tries to maximize its single period payoff. The primary user acts to maximize its total (undiscounted)

Note that $e^{cx} = 1 + cx + c^2x^2/2 + e(x)$, where $e(x) = o(x^2)$. The result follows by substituting this Taylor expansion in the above expression for \tilde{g}_{12} , and considering the cases $c < 1$, $c > 1$, and $c = 1$, respectively.

payoff over the entire horizon. Even though the secondary user is myopic, it has *perfect* recall of the actions taken by both the primary and the secondary user in previous periods of the game. We find that this model can be studied using seminal results from the economic literature on *reputation effects*; in particular, our main insight is that the primary may choose to spread power against an entering secondary, even if it is not profitable in a single period to do so—the goal being to “scare” the secondary into never entering again. We first describe our repeated game model in section 4.1. Then, in section 4.2, we analyze sequential equilibria for such games.

4.1 A Repeated SBGI-E Game

We first assume that at the beginning of the repeated game, nature chooses (independent) cross channel gains g_{12} and g_{21} from a known common prior distribution F , and these channel gains stay constant for the entire duration of the game. As in the preceding section, we assume $g_{11} = g_{22} = 1$, to isolate the effect of interference. We continue to make the assumption (for technical simplicity) that the secondary user is not aware of g_{21} , while the primary user knows the value of g_{12} ; insights for the case where the primary does not know g_{12} are offered in the appendix. As before, the channel gains determine the data rate obtained by each user.

We assume that the primary and the secondary user play the same SBGI-E game in each period; i.e., each period the secondary decides whether to enter or exit. Because the secondary player is assumed to be myopic, once the secondary player enters the game its best response to the primary user’s action is uniquely defined by Proposition 1. In particular, post-entry, the best response of the secondary is *identical* to the action taken by the primary—regardless of the channel gains. Thus, we can reduce the three-stage SBGI-E game into a two-stage SBGI-E game. In the first stage of this reduced game, the secondary user chooses either N (enter) or X (exit). If the secondary user enters, then in the second stage, the primary user chooses either SH (share) or SP (spread). The payoffs of the players are then realized, using the fact that the post-entry action of the secondary user is same as the action of the primary user.

We let $(a_{1,t}, a_{2,t})$ denote the actions chosen by the two players at each time period t . If the secondary exits, then the primary is always strictly better off spreading power across subchannels instead of concentrating in a single subchannel. Thus $a_{2,t} = X$ is never followed by $a_{1,t} = SH$ in equilibrium, so without loss of generality we assume $(a_{1,t}, a_{2,t}) \in \{(SP, X), (SH, N), (SP, N)\}$.

We assume both users have *perfect* recall of the actions taken by the users in previous peri-

ods. Let $h_{i,t}$ denote the *history* recalled by player i in period t . Then, $h_{1,t} = (\mathbf{a}_1, \dots, \mathbf{a}_{t-1}, a_{2,t})$ and $h_{2,t} = (\mathbf{a}_1, \dots, \mathbf{a}_{t-1})$ (since the primary observes the entry decision of the secondary before moving). The strategy $s_i(h_{i,t})$ of a user i is a probability distribution over available actions ($\{X, N\}$ for the secondary user, and $\{SH, SP\}$ for the primary user). Note in particular that in this section we allow *mixed strategies* for both players.

By a slight abuse of notation, let $\Pi_i(\mathbf{a}_t)$ denote the payoff of player i in period t . For the primary user, the payoff in each period is the maximum data rate it gets in that period, and its objective is to maximize the total payoff $\sum_{t=1}^T \Pi_1(\mathbf{a}_t)$. Here T is the length of the horizon for the repeated game. In each period t , let Π_0 be the payoff obtained by the primary user if the secondary user exits; then $\Pi_0 = \Pi_1(SP, X) = \log\left(1 + \frac{P}{2N_0}\right)$. As before, we do not assume a cost of power for the primary user; this does not affect the results presented in this section. On the other hand, the secondary user is considered to be myopic: its objective is to maximize its one period payoff. If the secondary user decides to exit the game, it obtains zero rate with no cost of power, so $\Pi_2(SP, X) = 0$.

The per-period payoffs of the primary player and the secondary player are thus given as:

$$\Pi_1(\mathbf{a}_t) = \begin{cases} \Pi_0, & \text{if } \mathbf{a}_t = (SP, X) \\ \Pi_1^{\text{share}}, & \text{if } \mathbf{a}_t = (SH, N) \\ \Pi_1^{\text{spread}}, & \text{if } \mathbf{a}_t = (SP, N); \end{cases} \quad \Pi_2(\mathbf{a}_t) = \begin{cases} 0, & \text{if } \mathbf{a}_t = (SP, X) \\ \Pi_2^{\text{share}} - kP, & \text{if } \mathbf{a}_t = (SH, N) \\ \Pi_2^{\text{spread}} - kP, & \text{if } \mathbf{a}_t = (SP, N). \end{cases} \quad (9)$$

Here Π_i^{share} and Π_i^{spread} are defined in (6). Since Π_i^{spread} may be stochastic, $\Pi_i(\mathbf{a}_t)$ may be stochastic as well.

4.2 Sequential Equilibrium of the Repeated Game

In this section we study sequential equilibria of the repeated SBGI-E game. Note that all exogenous parameters are known by the primary under Assumption 1. However, the secondary does not know the channel gain g_{21} of the primary, and instead maintains a conditional distribution, or *belief*, of the value of g_{21} given the observed history $h_{2,t}$. The secondary user updates his beliefs in a Bayesian manner as the history evolves (see footnote 4).

As shown in Proposition 3, if $g_{12} < \tilde{g}_{12}$ for the single period SBGI-E game, the secondary user's entry decision depends only on its realized channel gain g_{12} and the cost of power. It follows that regardless of the secondary's beliefs, if $g_{12} < \tilde{g}_{12}$ the secondary user either enters in every period or it stays out in every period; i.e., its strategy is independent of history. Thus, in the sequential equilibrium of the repeated game, each period follows the sequential

equilibrium of the single period game (cf. Proposition 3).

More interesting behavior arises if $g_{12} > \tilde{g}_{12}$. In this case, from Proposition 3, the secondary user prefers to enter if there is high probability the subchannels will be shared by the primary, and prefers to exit otherwise. Since the primary's action depends on its gain g_{21} , in this case we must calculate the secondary user's conditional distribution of g_{21} after each history $h_{2,t}$. Belief updating can lead to significant analytical complexity, as the belief is infinite-dimensional (a distribution over a continuous space).

However, the result of Proposition 3 suggests perhaps some reduction may be possible: as noted there, the secondary's action only depends on its belief about whether g_{21} is larger or smaller than g^* , which can be reduced to a scalar probability. If we can exhibit a sequential equilibrium of the repeated game in which the primary's action only depends on whether g_{21} is larger or smaller than g^* as well, then we can represent the secondary's belief by a scalar sufficient statistic, namely the probability that g_{21} is larger than g^* .

Remarkably, we show that precisely such a reduction is possible, by exhibiting a sequential equilibrium with the desired property. Define $\mu_{2,t}(h_{2,t}) = \mathcal{P}(g_{21} < g_{21}^* | h_{2,t})$. We exhibit a sequential equilibrium where (1) the entry decision of the secondary user in period t is based only on this probability; and (2) the primary's strategy is entirely determined by whether g_{21} is larger or smaller than g^* . In this equilibrium μ_t will be a sufficient statistic for the history of the play until time t .

The equilibrium we exhibit has the property that the primary user can exploit the lack of knowledge of the secondary user. To illustrate this point, consider a simplistic 2 period game. Assume that $g_{21} > g^*$, so that in a single period game the primary prefers SH to SP after entry by the secondary. If the secondary enters in the first period and the primary plays SP (spread), the secondary may mistakenly believe g_{21} to be *small*—and thus expect its payoff to be negative in the second period as well, and hence not enter. The primary thus obtains total payoff $\Pi_1^{\text{spread}} + \Pi_0$. By contrast, if the primary had shared in the first period, the secondary would certainly have entered in the second period as well, and in this case the primary obtains payoff $2\Pi_1^{\text{share}}$. It can be easily shown that if $g_{21} < 1$, then $\Pi_1^{\text{spread}} + \Pi_0 > 2\Pi_1^{\text{share}}$. Thus under such conditions, the primary user can benefit by spreading even though its single-period payoff is maximized by sharing.

The above example highlights the fact that the secondary user's lack of information can be exploited by the primary user for its own advantage. The primary user can masquerade and build a *reputation* as an “aggressive” player, thereby preventing entry by the secondary user. Such “reputation effects” were first studied in the economics literature [7, 11], where

the authors show that the lack of complete information can lead to such effects.⁷

For the repeated SGI-E game with $g_{12} > \tilde{g}_{12}$, a sequential equilibrium can be derived using an analysis closely following [7]. Here (for notational simplicity) the periods are numbered in reverse numerical order. Thus, T denotes the first chronological period, and 1 the last. We have the following theorem.

Theorem 5. *Suppose Assumptions 1 and 2 hold, and that $g_{12} > \tilde{g}_{12}$. Also assume that $\mathcal{P}(g_{21} < 1) = 1$. Let d be defined as in (8). Then the following actions and the belief update rule form a sequential equilibrium of the finite horizon repeated SBGI-E game:*

1. *The secondary user in period t enters the system if $\mu_{2,t}(h_{2,t}) < d^t$, and it exits the system if $\mu_{2,t}(h_{2,t}) > d^t$. If $\mu_{2,t} = d^t$, the secondary user enters with probability λ , and exits with probability $1 - \lambda$, where:*

$$\lambda = 2 - \frac{\Pi_0 - \Pi_1^{spread}}{\Pi_0 - \Pi_1^{share}}.$$

2. *If $g_{21} < g^*$, the primary user always spreads its power. If $g_{21} > g^*$, then after entry by a secondary user in period $t > 1$, the primary user always spreads if $\mu_t \geq d^{t-1}$, and otherwise randomizes, with the probability of spreading equal to:*

$$\gamma = \frac{\mu_t}{(1 - \mu_t)} \frac{(1 - d^{t-1})}{d^{t-1}}. \quad (10)$$

For $t = 1$, a primary user with $g_{21} > g^$ always shares.*

3. *The beliefs μ_t are updated as follows:*

$$\mu_{2,t}(h_{2,t}) = \begin{cases} \mu_{2,t+1}(h_{2,t+1}), & \text{if } a_{t+1} = (X, \phi); \\ \max\{d^t, \mu_{2,t+1}(h_{2,t+1})\}, & \text{if } a_{t+1} = (N, SP) \text{ and } \mu_{2,t+1}(h_{2,t+1}) > 0; \\ 0, & \text{if } a_{t+1} = (N, SH) \text{ or } \mu_{2,t+1}(h_{2,t+1}) = 0. \end{cases} \quad (11)$$

The proof of the above theorem follows steps similar to those in [7]. An outline of the proof for $T = 2$ is given in the appendix. Note that $d < 1$, so d^t increases as the game progresses (since $t = T, \dots, 1$). The first period in which the secondary will enter is when its initial belief $\rho = \mathcal{P}(g_{21} < g^*)$ first falls below d^t ; thus even if $g_{21} > g^*$, entry is deterred

⁷The lack of information can also be used to sustain desirable equilibria, as shown in the case of the finitely repeated prisoner's dilemma [6]. For a comprehensive treatment of such reputation effects see [10].

from T up to (approximately) $t^* = \frac{\log(\rho)}{\log(d)}$. It is important to note that this *never* happens in a complete information game: if the secondary knew $g_{21} > g^*$ it would enter in *every* time period, and the primary would always share.

We conclude by noting that it is straightforward to show that equilibria may be inefficient: for fixed ρ and d , t^* is constant, so as T increases the number of periods in which entry is deterred increases without bound. For parameter values where it would have been better to allow both users to transmit in each period, the resulting equilibrium is clearly inefficient.

5 Conclusion

We have studied distributed resource allocation in wireless systems via static and sequential Gaussian interference games of incomplete information. Our analysis shows that equilibria of these Bayesian games exhibit significant differences from their complete information counterparts. In particular, we have shown in two settings that static Gaussian interference games have a unique, potentially inefficient equilibrium where all users spread their powers. More dramatically, in repeated sequential games, we have shown that the lack of channel information can lead to reputation effects. Here the primary user has an incentive to alter its power profile to keep incoming secondary users from entering the system.

A Sequential Interference Games with Two-Sided Uncertainty

The model for the two stage sequential Bayesian Gaussian interference (SBGI) game described in Section 3.1 assumed that the primary user is aware of the channel gain g_{12} . In this appendix we analyze the case where both users are aware of only their own incident channel gains. We refer to this case as *two-sided uncertainty*. In Section A.1, we analyze a two stage sequential game analogous to Section 3.1. In Section A.2, we extend this analysis to include an entry stage as well.

A.1 A Two Stage Sequential Game

As before, we solve for the equilibrium path using backward induction. Whenever the primary user chooses SP , the best response of the secondary user is to choose SP regardless of the value of g_{12} ; the reasoning is identical to the proof of Lemma 4. Therefore, if the

primary user chooses the action SP , its payoff is Π_1^{spread} which is given by

$$\Pi_1^{\text{spread}}(g_{21}) = \log \left(1 + \frac{P/2}{N_0 + g_{21}P/2} \right). \quad (12)$$

(Note that we now explicitly emphasize the dependence of the payoff on the channel gain g_{21} .)

If the primary user decides to share the bandwidth, i.e., chooses the action SH , then depending upon the value of g_{12} the secondary user will choose between share (SH) or spread (SP). From Lemma 4 we know that if $g_{12} < 1/2$, the secondary user would prefer to spread its power even though the primary user shares the subchannels. Let $\kappa = \mathcal{P}(g_{12} < 1/2)$. Since the primary user is unaware of g_{12} , its expected payoff (denoted by $\bar{\Pi}_1^{\text{share}}$) if it shares the bandwidth is given by

$$\bar{\Pi}_1^{\text{share}}(g_{21}) = \frac{1-\kappa}{2} \log \left(1 + \frac{P}{N_0} \right) + \frac{\kappa}{2} \log \left(1 + \frac{P}{N_0 + g_{21}P/2} \right) \quad (13)$$

We have the following proposition.

Proposition 4. *In the first stage of the SBGI game with two-sided uncertainty, there exists a threshold $\hat{g}_{21}(\kappa) > 0$ (possibly infinite) such that if $g_{21}(\kappa) < \hat{g}_{21}$, the primary user always spreads its power, and if $g_{21} > \hat{g}_{21}(\kappa)$, the primary user always shares the subchannels.*

Proof. To decide whether to spread or share, the primary user needs to compare $\Pi_1^{\text{spread}}(g_{21})$ to $\bar{\Pi}_1^{\text{share}}(g_{21})$. We begin by establishing that $\Delta(g_{21}) = \Pi_1^{\text{spread}}(g_{21}) - \bar{\Pi}_1^{\text{share}}(g_{21})$ is strictly decreasing in g_{21} . To see this, note that if we define $y(g_{21}) = P/(N_0 + g_{21}P/2)$, then

$$\Delta'(g_{21}) = y'(g_{21}) \left(\frac{1/2}{1 + y(g_{21})/2} - \frac{\kappa/2}{1 + y(g_{21})} \right).$$

Since $y'(g_{21}) < 0$, and $0 \leq \kappa \leq 1$, we conclude that $\Delta'(g_{21}) < 0$ for all g_{21} ; i.e., $\Delta(g_{21})$ is strictly decreasing in g_{21} , as required.

When $g_{21} = 0$, we have:

$$\begin{aligned} \bar{\Pi}_1^{\text{share}}(0) &= \frac{1}{2} \log \left(1 + \frac{P}{N_0} \right); \\ \Pi_1^{\text{spread}}(0) &= \log \left(1 + \frac{P}{2N_0} \right). \end{aligned}$$

Since $\log(1+x)$ is a strictly concave function of x , $\Pi_1^{\text{spread}}(0) > \bar{\Pi}_1^{\text{share}}(0)$. However, if g_{21} is

large, the payoffs to the primary user in the two cases are given by:

$$\begin{aligned}\lim_{g_{21} \rightarrow \infty} \bar{\Pi}_1^{\text{share}} &= \frac{1-\kappa}{2} \log \left(1 + \frac{P}{N_0} \right); \\ \lim_{g_{21} \rightarrow \infty} \Pi_1^{\text{spread}} &= 0.\end{aligned}$$

Thus, $\lim_{g_{21} \rightarrow \infty} \bar{\Pi}_1^{\text{share}} \geq \lim_{g_{21} \rightarrow \infty} \Pi_1^{\text{spread}}$. If $\kappa = 1$, then $\Delta(g_{21}) > 0$ for all g_{21} , and thus user 1 always spreads; i.e., $\hat{g}_{21}(\kappa) = \infty$. Otherwise, there exists a unique finite threshold $\hat{g}_{21}(\kappa) > 0$ determined by the equation $\Delta(g_{21}) = 0$, as required. \square

For the secondary user, the best response is to spread the power if the primary user spreads its power. However, if the primary user decides to share the subchannels, the secondary user will share if $g_{12} > 1/2$; otherwise it spreads its power. This completely determines the sequential equilibrium for the SBGI game with two-sided uncertainty.

A.2 A Sequential Game with Entry

We now consider the sequential Bayesian Gaussian interference with entry (SBGI-E) game when both users are only aware of their own incident channel gains; i.e., player i only knows $g_{-i,i}$. As before, the secondary user first decides whether to enter or not; if the secondary user exits, then the primary uses the entire band without competition. If the secondary enters, then play proceeds as in the SBGI game of the preceding section. Further, if the secondary chooses to enter, it incurs a cost of power denoted by kP .

To describe a sequential equilibrium for the SBGI-E game with two-sided uncertainty, we define $\Pi_2^{(\text{share}, a)}$ to be the rate of the secondary user when the primary user shares the subchannels and the secondary user chooses the action $a \in \{SH, SP\}$. Thus, we have:

$$\Pi_2^{(\text{share}, \text{share})} = \Pi_2^{\text{share}} = \frac{1}{2} \log \left(1 + \frac{P}{N_0} \right), \quad (14)$$

$$\Pi_2^{(\text{share}, \text{spread})} = \frac{1}{2} \log \left(1 + \frac{P}{2N_0} \right) + \frac{1}{2} \log \left(1 + \frac{P/2}{N_0 + g_{12}P} \right). \quad (15)$$

We also recall the rate to the secondary user when both users spread their powers, denoted Π_2^{spread} (cf. (6)):

$$\Pi_2^{\text{spread}} = \log \left(1 + \frac{P/2}{N_0 + g_{12}P/2} \right).$$

In this game, we must be particularly careful about how uncertainty affects sequential decisions. In particular, since the entry decision of the secondary user will depend on the gain

g_{12} , the primary user *learns* about the value of g_{12} from the initial action of the secondary; this is modeled through the updated *belief* of the primary user, i.e., its conditional distribution of g_{12} given the initial action of the secondary. Post-entry, the play proceeds as in the SBGI game with two-sided uncertainty considered in the previous section.

The following proposition formally describes sequential equilibria for the SBGI-E game with two-sided uncertainty; for simplicity, we assume the gain distributions have full support on $(0, \infty)$, but this is unnecessary.

Proposition 5. *Assume that the channel gains g_{12} and g_{21} have strictly positive densities on $(0, \infty)$. For the sequential Bayesian Gaussian interference game with entry (SBGI-E), any sequential equilibrium consists of threshold strategies for both the primary and secondary user; i.e., there exists a threshold \hat{g}_{12} such that the secondary user enters if $g_{12} < \hat{g}_{12}$, and exits if $g_{12} > \hat{g}_{12}$; and post-entry, there exists a $\hat{\kappa}$ such that the primary user spreads if $g_{21} < \hat{g}_{21}(\hat{\kappa})$, and shares if $g_{21} > \hat{g}_{21}(\hat{\kappa})$ (where $\hat{g}_{21}(\cdot)$ is the threshold of Proposition 4). In the third stage, if the primary user spreads its power, the secondary user also spreads its power regardless of the value of g_{12} . However, if the primary user shares the subchannels, the secondary user spreads its power if $g_{12} < 1/2$ and it shares otherwise.*

The belief $\hat{\kappa}$ is computed using Bayes' rule:

$$\hat{\kappa} = \frac{\kappa \mathcal{P}(N \mid g_{12} < 1/2)}{\kappa \mathcal{P}(N \mid g_{12} < 1/2) + (1 - \kappa) \mathcal{P}(N \mid g_{12} > 1/2)}, \quad (16)$$

where N denotes the “entry” action of the secondary user, and $\kappa = P(g_{12} < 1/2)$ (the initial belief of the primary).

Proof. We first show that in equilibrium, the primary must have a threshold strategy of the form specified. Note that if the secondary player exits in the first stage, the primary user has no action to take and the game ends. If the secondary player enters, the primary user updates its belief about g_{12} via Bayes' rule as given in (16), given the entry strategy of the secondary. Here the probabilities are with respect to the uncertainty in g_{12} .

Now suppose that in equilibrium, the post-entry belief of the primary user is fixed as $\hat{\kappa}$. We consider the entry decision of the secondary user. The secondary user's decision between entry or exit depends on the post-entry action taken by the primary user. From Proposition 4, we know that post-entry, the primary user will spread (resp., share) if g_{21} is less than (resp., greater than) the threshold $\hat{g}_{21}(\hat{\kappa})$. (Here the threshold \hat{g}_{21} depends on the post-entry belief $\hat{\kappa}$ of the primary user.) Thus, the decision taken by the secondary user in the first stage depends on its initial belief $\alpha = \mathcal{P}(g_{21} < \hat{g}_{21}(\hat{\kappa}))$. Since $\hat{g}_{21}(\hat{\kappa}) > 0$ from Proposition 4, it follows that $\alpha > 0$.

After entry, with probability α , the primary user spreads its power; since the best response of the secondary user to the spreading action by the primary user is to also spread, the payoff of the secondary user in this case is Π_2^{spread} . With probability $1 - \alpha$, the primary user shares the subchannels. Conditioned on the sharing action by the primary user, the secondary user will spread its power if $g_{12} < 1/2$ and will share otherwise. Thus, the expected payoff of the secondary user upon entry is given by the function $h(g_{12}, \alpha)$, defined as follows:

$$h(g_{12}, \alpha) = \begin{cases} \alpha \Pi_2^{\text{spread}} + (1 - \alpha) \Pi_2^{(\text{share}, \text{spread})}, & \text{if } g_{12} \leq 1/2, \\ \alpha \Pi_2^{\text{spread}} + (1 - \alpha) \Pi_2^{(\text{share}, \text{share})}, & \text{if } g_{12} > 1/2. \end{cases} \quad (17)$$

It is easy to check that $h(g_{12}, \alpha)$ is continuous; further, for fixed $\alpha > 0$, $h(g_{12}, \alpha)$ is a strictly decreasing function of g_{12} . The secondary user will enter if its expected payoff $h(g_{12}, \hat{\alpha})$ is greater than its cost of power kP . If, $h(0, \alpha) < kP$, let $\hat{g}_{12} = 0$. Similarly, if $\lim_{g_{12} \rightarrow \infty} h(g_{12}, \alpha) > kP$, we define $\hat{g}_{12}(\alpha) = \infty$. Otherwise there exists a unique value of $\hat{g}_{12} \in (0, \infty)$ with $h(\hat{g}_{12}, \alpha) = kP$. Thus, the secondary user will enter if $g_{12} < \hat{g}_{12}$, and exit if $g_{12} > \hat{g}_{12}$. This concludes the proof. \square

B Sequential Equilibrium for a Two Period Repeated Game

In this appendix, for completeness we give an outline of the proof of Theorem 5 for the two period repeated SBGI-E game with *single-sided uncertainty*. The arguments given below are based on those given in [7], and we refer the reader to that paper for details and extensions.

To analyze the two period repeated SBGI-E game, we use backward induction. We number the periods in reverse numerical order. Thus, period 1 is the last period of the game, and period T is the first period of the game; period t follows period $t + 1$. To specify the equilibrium, we need to specify the actions of the secondary user and the primary user in all periods and after all possible values of histories and beliefs. Note that if $g_{21} < g^*$, the primary user always spreads its power. So we need to specify the action of the primary user for the case when $g_{21} > g^*$; we refer to this type of primary user as a *high-gain primary user*. For the remainder of this discussion, we only specify the actions of the high-gain primary user. (We ignore the $g_{21} = g^*$ case since channel gains have continuous densities). Also note that although the beliefs of the secondary user are history dependent, we suppress the history dependence of the beliefs for notational simplicity.

- **Period 1:** In period 1 (which is the last period), if the secondary user does not enter,

the high-gain primary user has no action to take. However, if the secondary user decides to enter, the high-gain primary user will share the subchannels, since sharing the subchannels is the best response of the high-gain primary user to an entry by the secondary user. To decide between entry and exit, the secondary user takes into account its belief $\mu_{2,1}$ about the channel gain g_{21} of the primary user. The secondary user will enter if its expected payoff in period 1 is greater than its payoff if it exits. This happens if

$$\mu_{2,1}(\Pi_2^{\text{spread}} - kP) + (1 - \mu_{2,1})(\Pi_2^{\text{share}} - kP) > 0 \implies \mu_{2,1} < \frac{\Pi_2^{\text{share}} - kP}{\Pi_2^{\text{share}} - \Pi_2^{\text{spread}}} = d. \quad (18)$$

Here d is defined as in (8). Thus, in equilibrium the secondary user enters (N) if its current belief $\mu_{2,1} < d$, it exits the system if $\mu_{2,1} > d$, and it is indifferent if $\mu_{2,1} = d$.

To find the current belief $\mu_{2,1}$, the secondary user observes the history of the play. If in period 2, the secondary user exits the game (X), no new information about the primary user is learned. Thus if $h = (X)$, we have $\mu_{2,1} = \mu_{2,2}$. Since at period 2, no history has been observed, we have $\mu_{2,2} = \rho$, which is the initial belief of the secondary user. However, if the secondary user in period 2 enters, the belief of the secondary user in period 1 would depend upon the action taken by the primary user in period 2. If the primary user shares the subchannels in period 1, then it is certain that $g_{21} > g^*$ and hence $\mu_{2,1} = 0$.

When the history (N, SP) is observed, the secondary user uses Bayes' rule to update its belief. Let γ denote the probability that the primary user would spread its power even if $g_{21} > g^*$. Then the total probability that the primary user would spread (in period 2) is $\mu_{2,2} + (1 - \mu_{2,2})\gamma$. Bayes' rule then implies that the belief in period 1 is given as

$$\mu_{2,1} = \frac{\mu_{2,2}}{\mu_{2,2} + (1 - \mu_{2,2})\gamma}. \quad (19)$$

Here the numerator is the probability that $g_{21} < g^*$ in period 2.

- **Period 2:** For the high-gain primary user in period 2, the action it takes in this period determines the history for period 1, and hence the action taken by the secondary user. The high-gain primary user thus needs to conjecture the behavior of the secondary user in period 1 to decide its action. Note that the belief of the secondary user in period 2

is same as the initial belief, i.e., $\mu_{2,2} = \rho$.

If the secondary user does not enter the game at this period, the primary user has no action to take. However, if the secondary user enters, the high-gain primary user has to choose between the actions SH or SP . It chooses this action so as to maximize its expected total payoff in the two periods. Here the expectation is over the randomness in the action taken by the secondary user in period 1⁸.

First note that in equilibrium $\gamma > 0$. To see this, let us assume otherwise, i.e., $\gamma = 0$. This implies that in equilibrium, if the secondary user enters, the high-gain primary user does not spread. Then the high-gain primary user's total payoff in 2 periods is $2\Pi_1^{\text{share}}$. However, if the high-gain primary user spreads in period 2, then the secondary user in period 1 has $\mu_{2,1} = 1$ (see (19)) and hence it does not enter. In this case, the total payoff to the high-gain primary user is $\Pi_1^{\text{spread}} + \Pi_0$ which is greater than $2\Pi_1^{\text{share}}$ since $g_{21} < 1$. Hence the high-gain primary user has a profitable deviation in equilibrium with $\gamma = 0$. Thus, in equilibrium $\gamma > 0$.

We consider two different cases. First suppose that $\mu_{2,2} = \rho \geq d$. In this case, regardless of the strategy of the primary user in period 2, we have

$$\mu_{2,1} > \rho \geq d.$$

Hence, the secondary user in period 1 would not enter after seeing the history of SP . So if the high-gain primary user in period 2 takes the action SP , the total payoff is $\Pi_1^{\text{spread}} + \Pi_0$. On the other hand taking the action SH would cause the secondary user in period 1 to enter, and hence the total payoff would be $2\Pi_1^{\text{share}}$. Since $\Pi_1^{\text{spread}} + \Pi_0 > 2\Pi_1^{\text{share}}$, the best response for the high-gain primary user in period 2 (if the secondary enters and $\rho > d$) is to spread the power.

The second case is when $\mu_{2,2} = \rho < d$. In this case, we first note that $\gamma < 1$. If we assume that $\gamma = 1$, then $\mu_{2,1} = \mu_{2,2} < d$ and hence the secondary user in period 1 would always enter and the high-gain primary would spread (since $\gamma = 1$). But we know that in period 1, the best response of the high-gain primary user to an entry is to share. Hence $\gamma < 1$. This implies that if $\mu_{2,2} < d$, then $0 < \gamma < 1$. Thus the high-gain primary user randomizes its policy over SP and SH . This is only possible if the secondary user in period 1 also randomizes over entry and exit. Let us denote the probability of secondary user entering in period 1 under these conditions as λ . Since

⁸We are assuming that secondary user 2 has already entered, so the only unknown factor is the action of the secondary user in period 1.

the high-gain primary user in period 2 is indifferent between spreading and sharing its expected payoff in both cases is the same. This gives

$$\begin{aligned}\Pi_1^{\text{spread}} + \lambda (\Pi_1^{\text{share}}) + (1 - \lambda)\Pi_0 &= 2\Pi_1^{\text{share}} \\ \implies \lambda &= 2 - \frac{\Pi_0 - \Pi_1^{\text{spread}}}{\Pi_0 - \Pi_1^{\text{share}}}.\end{aligned}$$

Also, since the secondary user in period 1 is indifferent between entry and exit, its belief is $\mu_{2,1} = d$. This gives

$$\frac{\mu_{2,2}}{\mu_{2,2} + (1 - \mu_{2,2})\gamma} = d \implies \gamma = \frac{\mu_{2,2}}{(1 - \mu_{2,2})} \frac{1 - d}{d}.$$

To determine the action of secondary user at period 2, we note that if it exits its payoff is 0. Now if $\mu_{2,2} = \rho \geq d$, the high-gain primary user spreads with probability 1, hence the best response for the secondary user is to exit. However, if $\mu_{2,2} = \rho < d$, then the secondary user's expected payoff is

$$(\Pi_2^{\text{spread}})(\rho + (1 - \rho)\gamma) + (1 - \rho)(1 - \gamma)\Pi_2^{\text{share}}.$$

If the above expected payoff is less than 0, the secondary user does not enter. Using the value of γ from above we get that if $\rho > d^2$, the secondary user exits. However, if $\rho < d^2$, the secondary user enters, and at equality the secondary user is indifferent. This completely specifies the sequential equilibrium for the two period repeated SBGI-E game.

The extension to an arbitrary finite horizon repeated game is similar to the arguments given above and we refer the reader to [7] for the detailed proof.

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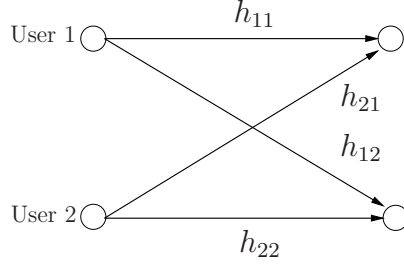


Figure 1: Two User Interference Channel

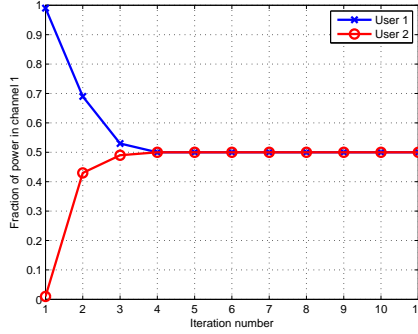
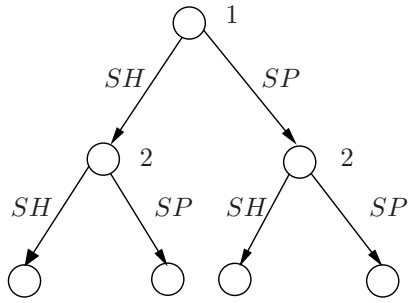
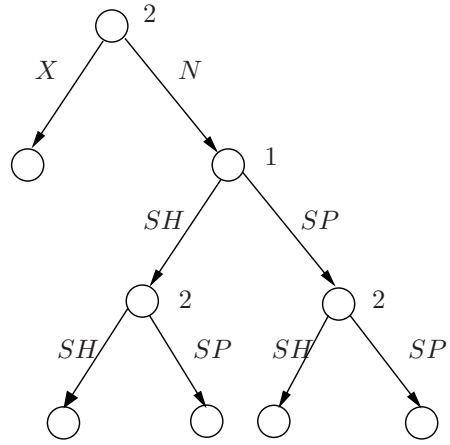


Figure 2: *Convergence of best response dynamics for two user UC-GI game.* We consider a model with $B = 2$ sub channels, $P = 1$, $N_0 = 0.01$, and we normalize $g_{11} = g_{22} = 1$. We assume g_{12} and g_{21} are both drawn from a uniform distribution on $[0, 1]$. We initiate the best response dynamics at $P_{11} = 1 - P_{12} = P$, and $P_{21} = 1 - P_{22} = 0$; observe that the powers P_{ic} converge to $P/2$ for each i and c . The behavior is symmetric if we instead initiate with $P_{11} = 1 - P_{12} = 0$, and $P_{21} = 1 - P_{22} = P$.



(a)



(b)

Figure 3: *Game trees for sequential games.* Player 1 is the *primary* user; player 2 is the *secondary* user. The tree in (a) describes the SBGI game. The tree in (b) describes the SBGI-E game.